

UNIVERSAL PORTFOLIOS IN STOCHASTIC PORTFOLIO THEORY

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ABSTRACT. Consider a family of portfolio strategies with the aim of achieving the asymptotic growth rate of the best one. Cover's solution is to build a wealth-weighted average which can be regarded as a buy-and-hold portfolio of portfolios. When an optimal portfolio exists, the wealth-weighted average converges to it by concentration of wealth. Under suitable conditions, we show that the distribution of wealth in the family satisfies a pathwise large deviation principle as time tends to infinity. In particular, we extend Cover's portfolio to the nonparametric family of functionally generated portfolios in stochastic portfolio theory and establish its asymptotic universality.

1. INTRODUCTION

The problem of portfolio selection is to decide, at each point in time, the distribution of capital over the available assets in order to maximize future wealth. For portfolios without short selling, the distribution at time t is given by a *portfolio weight* vector $\pi(t) = (\pi_1(t), \dots, \pi_n(t))$ with non-negative components summing to 1, where n is the number of assets. Since Markowitz's seminal paper [Mar52] there has been an explosive growth of literature on the theory and practice of portfolio selection. The mainstream approach, due to Markowitz, consists of two major steps. First we estimate a model of the joint distribution of future asset returns (usually specified in terms of the first and second moments). Then, based on the investor's preference and risk aversion (described by a utility function), we find the portfolio weights representing the optimal trade-off between risk and return. We refer the reader to [CK06] for mathematical details as well as practical considerations.

The above approach depends on the investor's non-observable preferences and requires forecasts of returns and risks. Can we construct good portfolios *without* assuming specific models of preferences and asset prices? In recent years two *model-free* approaches emerged which attempt to achieve this goal. Among other things, a model-free approach requires less assumptions and is thus more robust.

1.1. Stochastic portfolio theory. Stochastic portfolio theory, first developed by Fernholz [Fer02] and extended by Fernholz and Karatzas [FK09] and others, is a *descriptive* theory of equity market and portfolio selection. By this we mean portfolio selection, as well as analysis of portfolio performance, can be based solely on *directly observable* quantities of the market, and under realistic conditions certain portfolios can be shown to outperform the market over sufficiently long horizons.

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To explain this more precisely let us introduce some notations. In an equity market with n stocks, let $X_i(t) > 0$ be the market capitalization of stock i at time t . The *market weight* of stock i is the ratio

$$(1.1) \quad \mu_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}.$$

The market weights are the portfolio weights of the *market portfolio*. It is a buy-and-hold portfolio representing the overall performance of the market. Suppose we arrange the market weights in descending order:

$$(1.2) \quad \mu_{(1)}(t) \geq \cdots \geq \mu_{(n)}(t).$$

Here the $\mu_{(k)}(t)$'s are the reverse order statistics, and the vector $(\mu_{(1)}(t), \dots, \mu_{(n)}(t))$ of ranked market weights is called the *capital distribution* of the market. It was observed (see [Fer02, Chapter 4]) that despite price and economic fluctuations, the distribution of capital exhibits remarkable stability over long periods. In particular, the equity market has remained *diverse*: the maximum market weight $\max_{1 \leq i \leq n} \mu_i(t)$ has been bounded away from 1. Moreover, the market appears to possess *sufficient volatility*: if one plots the cumulative realized volatility of $\mu(t)$, its rate of increase is bounded below. Under these and only these descriptive conditions, there exist portfolios that are guaranteed to outperform the market portfolio over sufficiently long horizons. For precise statements and their relationship with the classical notion of arbitrage, see [FK09, Chapter 2]. These so-called *functionally generated portfolios* are deterministic functions of the current market weights; in particular, they require no dynamic forecast of asset prices and optimization of portfolio weights. A classic example is the *entropy-weighted portfolio* whose portfolio weights are given by

$$\pi_i(t) = \frac{-\mu_i(t) \log \mu_i(t)}{\sum_{j=1}^n -\mu_j(t) \log \mu_j(t)}.$$

In [PW15] we established an elegant connection between functionally generated portfolio, convex analysis and optimal transport. Moreover, we showed that functionally generated portfolios are, in a certain sense, the only portfolio weight functions capable of beating the market, in the long run, under the conditions of diversity and sufficient volatility.

1.2. Universal portfolio theory. Universal portfolio theory is a very active research field in mathematical finance and machine learning. Instead of giving an extensive review (which we refer the reader to the recent survey [LH14] and the references therein) let us explain the main ideas of Cover's classic paper [Cov91] which started the subject. A portfolio over n stocks is said to be *constant-weighted*, or *constantly rebalanced*, if the portfolio weights $\pi(t) \equiv \pi$ are constant over time. Historically, it has been observed that a rebalanced portfolio frequently outperforms a buy-and-hold portfolio of the constituent stocks (see [PW13] for a theoretical justification). Let $V_\pi(t)$ be the wealth of the constant-weighted portfolio π at time t (with initial value $V_\pi(0) = 1$), where π ranges over the closed unit simplex

$$\overline{\Delta}_n = \left\{ p = (p_1, \dots, p_n) \in \mathbb{R}^n : p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}.$$

Cover asked the following question: Without *any* knowledge of future stock prices, is it possible to invest in such a way that the resulting wealth is close to

$$V^*(t) = \max_{\pi \in \bar{\Delta}_n} V_\pi(t),$$

the performance of the *best* constant-weighted portfolio chosen with hindsight? While this seems an unrealistically ambitious goal, Cover constructed a nonanticipative sequence of portfolio weights $\hat{\pi}(t)$ such that the resulting wealth $\hat{V}(t)$ satisfies the *universality property*

$$(1.3) \quad \frac{1}{t} \log \frac{\hat{V}(t)}{V^*(t)} \geq \frac{C}{t^{(n-1)/2}} \rightarrow 0,$$

where $C > 0$ is a constant, for *arbitrary* sequences of stock returns. Explicitly, *Cover's universal portfolio* is given by

$$(1.4) \quad \hat{\pi}(t) = \frac{\int_{\bar{\Delta}_n} \pi V_\pi(t) d\pi}{\int_{\bar{\Delta}_n} V_\pi(t) d\pi}.$$

That is, $\hat{\pi}(t)$ is the average of all constant-weighted portfolios weighted by their performances, and it can be shown that

$$\hat{V}(t) = \frac{\int_{\bar{\Delta}_n} V_\pi(t) d\pi}{\int_{\bar{\Delta}_n} d\pi}.$$

This allows us to view Cover's portfolio as a buy-and-hold portfolio of all constant-weighted portfolios, where each portfolio receives the same infinitesimal wealth initially. Cover's result (1.3) states that the maximum and average of $V_\pi(t)$ over $\pi \in \bar{\Delta}_n$ have the same asymptotic growth rate, and can be viewed as a consequence of Laplace's method of integration and the fact that for constant-weighted portfolios the map $\pi \mapsto V_\pi(t)$ is essentially a multiple of a normal density. While numerous alternative portfolio selection algorithms have been proposed for constant-weighted and other families of portfolios, the idea of forming a wealth-weighted average underlies many of these generalizations.

1.3. Summary of main results. It is natural to ask if functionally generated portfolios and Cover's universal portfolio are connected in some way [FK09, Remark 11.7]. Recently, [Bro14] showed that Cover's portfolio (1.4) is, in a generalized sense, functionally generated. With hindsight, this is not surprising since Cover's portfolio is a buy-and-hold portfolio of constant-weighted portfolios, and both buy-and-hold portfolios and constant-weighted portfolios are functionally generated [Fer02, Example 3.1.6]. Instead, it is more interesting to think of Cover's portfolio as a market portfolio where each constituent asset is the value process of a portfolio in a family. The capital distribution (1.2) then generalizes to the *distribution of wealth* over the portfolios. While the capital distribution of an equity market is typically stable and diverse, this is not true for the distribution of wealth over a family of portfolios. Quite the contrary, wealth often concentrates *exponentially* around an optimal portfolio, and under suitable conditions this can be quantified by a pathwise large deviation principle (LDP). Moreover, we show that Cover's portfolio (1.3) can be generalized to the nonparametric family of functionally generated portfolios which contains the constant-weighted portfolios.

To state the main results let us introduce informally some concepts which will be treated systematically in Section 2. We consider an equity market with $n \geq 2$

stocks in discrete time. The dynamics of the market is modeled by a sequence $\{\mu(t)\}_{t=0}^{\infty}$ of market weights with values in the open unit simplex Δ_n . We assume there is a constants $M > 0$ such that $\frac{1}{M} \leq \frac{\mu_i(t+1)}{\mu_i(t)} \leq M$ for all i and t (but the investor does not know the value of M). Consider a family $\{\pi_\theta\}_{\theta \in \Theta}$ of portfolio map maps, where Θ is a topological index set and each π_θ is a map from Δ_n to $\overline{\Delta}_n$. If the investor chooses the portfolio map π_θ , the portfolio weight vector at time t is given by $\pi_\theta(\mu(t))$ which depends only on $\mu(t)$. For convenience and following the tradition of stochastic portfolio theory, we measure the values of all portfolios relative to that of the market portfolio. Thus we define the *relative value* $V_\theta(t)$ of the self-financing portfolio π_θ by

$$(1.5) \quad V_\theta(0) = 0, \quad V_\theta(t+1) = V_\theta(t) \sum_{i=1}^n \pi_{\theta,i}(\mu(t)) \frac{\mu_i(t+1)}{\mu_i(t)}.$$

(See Definition 2.2). Imagine at time 0 we distribute wealth over the family according to a Borel probability measure ν_0 on Θ ; we call ν_0 the *initial distribution*. The *wealth distribution* of the family $\{\pi_\theta\}_{\theta \in \Theta}$ at time t is the Borel probability measure ν_t on Θ defined by

$$(1.6) \quad \nu_t(B) = \frac{1}{\int_{\Theta} V_\theta(t) d\nu_0(\theta)} \int_B V_\theta(t) d\nu_0(\theta), \quad B \subset \Theta.$$

We will be interested in situations where the wealth distribution of the family $\{\pi_\theta\}_{\theta \in \Theta}$ concentrates exponentially around some optimal portfolio. A natural way to quantify this is to prove a *large deviation principle* (LDP). A standard reference of large deviation theory is [DZ98].

Definition 1.1. Let $I : \Theta \rightarrow [0, \infty]$ be a lower-semicontinuous function, called the rate function. We say that the sequence $\{\nu_t\}_{t=0}^{\infty}$ satisfies the large deviation principle on Θ with rate I if the following statements hold.

(i) (Upper bound) For every closed set $F \subset \Theta$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \nu_t(F) \leq - \inf_{\theta \in F} I(\theta).$$

(ii) (Lower bound) For every open set $G \subset \Theta$,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \nu_t(G) \geq - \inf_{\theta \in G} I(\theta).$$

A sufficient condition for existence of LDP is that the *asymptotic growth rate*

$$(1.7) \quad W(\theta) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V_\theta(t)$$

exists for all $\theta \in \Theta$ and the map $\theta \mapsto V_\theta(t)$ is sufficiently smooth. As preparation, in Section 3 we study a simple situation where the family $\{\pi_\theta\}_{\theta \in \Theta}$, as maps from Δ_n to $\overline{\Delta}_n$, is *totally bounded* in the supremum metric.

Theorem 1.2. Let $\{\pi_\theta\}_{\theta \in \Theta}$ be a totally bounded family of portfolio maps from Δ_n to $\overline{\Delta}_n$. Suppose the asymptotic growth rate $W(\theta) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V_\theta(t)$ exists for all $\theta \in \Theta$ and the initial distribution ν_0 has full support on Θ . Then the sequence of wealth distribution satisfies LDP on Θ with rate function

$$I(\theta) = W^* - W(\theta),$$

where $W^* = \sup_{\theta \in \Theta} W(\theta)$.

In Section 4 we consider the family of functionally generated portfolios. Following [PW15] and [Won15], a portfolio map $\pi : \Delta_n \rightarrow \overline{\Delta}_n$ is said to be *functionally generated* if there exists a concave function $\Phi : \Delta_n \rightarrow (0, \infty)$ such that

$$(1.8) \quad \sum_{i=1}^n \pi_i(p) \frac{q_i}{p_i} \geq \frac{\Phi(q)}{\Phi(p)}$$

for all $p, q \in \Delta_n$; the function Φ is called the *generating function* of π . Geometrically, (1.8) means that the vector $\left(\frac{\pi_1(p)}{p_1}, \dots, \frac{\pi_n(p)}{p_n}\right)$ defines a supergradient of the concave function $\log \Phi$ at p . Conversely, any positive concave function on Δ_n generates a functionally generated portfolio. As an example, the constant-weighted portfolio (π_1, \dots, π_n) is generated by the geometric mean $\Phi(p) = p_1^{\pi_1} \cdots p_n^{\pi_n}$. We denote the family of functionally generated portfolios by \mathcal{FG} . We endow \mathcal{FG} with the topology of uniform convergence. It is clear that \mathcal{FG} is infinite dimensional and is thus ‘nonparametric’. It can be shown that \mathcal{FG} is convex.

Let

$$\mathbb{P}_t = \frac{1}{t} \sum_{s=0}^{t-1} \delta_{(\mu(s), \mu(s+1))}$$

be the empirical measure of the pair $(\mu(s), \mu(s+1))$ up to time t . In the following theorem we impose an asymptotic condition on $\{\mathbb{P}_t\}_{t=0}^\infty$ in the spirit of [Jam92].

Theorem 1.3. *Suppose \mathbb{P}_t converges weakly to an absolutely continuous Borel probability measure \mathbb{P} on $\Delta_n \times \Delta_n$.*

- (i) *(Glivenko-Cantelli property) The asymptotic growth rate $W(\pi)$ defined by (1.7) exists for all $\pi \in \mathcal{FG}$. Furthermore, we have*

$$\lim_{t \rightarrow \infty} \sup_{\pi \in \mathcal{FG}} \left| \frac{1}{t} \log V_\pi(t) - W(\pi) \right| = 0.$$

- (ii) *(LDP) Let ν_0 be any initial distribution on \mathcal{FG} . Then the sequence $\{\nu_t\}_{t=0}^\infty$ of wealth distributions given by (1.6) satisfies LDP with rate*

$$I(\pi) = \begin{cases} W^* - W(\pi) & \text{if } \pi \in \text{supp}(\nu_0), \\ \infty & \text{otherwise,} \end{cases}$$

where $W^* = \sup_{\pi \in \text{supp}(\nu_0)} W(\pi)$.

- (iii) *(Universality) There exists a probability distribution ν_0 on \mathcal{FG} such that $W^* = \sup_{\pi \in \text{supp}(\nu_0)} W(\pi) = \sup_{\pi \in \mathcal{FG}} W(\pi)$ for any absolutely continuous \mathbb{P} . For this initial distribution, consider Cover’s portfolio*

$$(1.9) \quad \hat{\pi}(t) = \int_{\mathcal{FG}} \pi(\mu(t)) d\nu_t(\pi).$$

Let $\hat{V}(t)$ be the value of this portfolio and let $V^*(t) = \sup_{\pi \in \mathcal{FG}} V_\pi(t)$. Then

$$(1.10) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \log \hat{V}(t) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V^*(t) = W^*.$$

In particular, we have $\lim_{t \rightarrow \infty} \frac{1}{t} \log \left(\hat{V}(t) / V^*(t) \right) = 0$.

In [Won15] we studied an optimization problem for functionally generated portfolio analogous to nonparametric density estimation. Regarding $\log V_\pi(t)$ as the log likelihood function for estimating π and ν_0 as the prior distribution, Theorem 1.3(ii)

shows that the posterior distribution ν_t satisfies an LDP. Convergence properties of posterior distributions in nonparametric statistics are delicate (see for example [BSW99]) and large deviation results are rare. For Dirichlet priors an LDP is proved in [GO00]. Theorem 1.3(iii) shows that the posterior mean (1.9) performs asymptotically as good as the best portfolio in \mathcal{FG} . If we think of the results in [Won15] as point estimation of functionally generated portfolio by maximum likelihood, Theorem 1.3 gives the Bayesian counterpart. For practical applications we would like to strengthen Theorem 1.3 to include quantitative bounds as well as algorithms for computing $\hat{\pi}$. This and other further problems are gathered in Section 5.

2. WEALTH DISTRIBUTIONS OF PORTFOLIOS

2.1. Stock and market weight. We consider an equity market with $n \geq 2$ stocks in discrete time. The dynamics of the market will be specified in terms of the *market weights* $\mu(t) = (\mu_1(t), \dots, \mu_n(t))$ given by (1.1). The vector of market weights $\mu(t)$ takes values in the open unit simplex Δ_n in \mathbb{R}^n . Suppose the market capitalization of stock i at time t is $X_i(t)$ and its simple return over the time interval $[t, t+1]$ is $R_i(t)$. The market weights at time $t+1$ are then given by

$$\mu_i(t+1) = \frac{X_i(t)(1+R_i(t))}{X_1(t)(1+R_1(t)) + \dots + X_n(t)(1+R_n(t))}.$$

We visualize the market as a discrete path in Δ_n . This includes only changes in capitalizations due to returns and excludes implicitly all changes due to corporate actions such as public offerings. We assume that $\{\mu(t)\}_{t=0}^\infty$ is any given sequence in Δ_n ; in particular, no underlying probability space is involved. The assumptions we state will be in terms of the path properties of the sequence $\{\mu(t)\}_{t=0}^\infty$. One such assumption is the following.

Assumption 2.1. There exists a constant $M > 0$ such that the market weight sequence $\{\mu(t)\}_{t=0}^\infty$ satisfies

$$(2.1) \quad \frac{1}{M} \leq \frac{\mu_i(t+1)}{\mu_i(t)} \leq M$$

for all $1 \leq i \leq n$ and $t \geq 0$. Let

$$(2.2) \quad \mathcal{S} = \left\{ (p, q) \in \Delta_n \times \Delta_n : \frac{1}{M} \leq \frac{q_i}{p_i} \leq M \text{ for } 1 \leq i \leq n \right\}.$$

Then (2.1) states that $(\mu(t), \mu(t+1)) \in \mathcal{S}$ for all $t \geq 0$.

Assumption 2.1 states that the relative returns of the stocks are bounded and is common in the literature (see for example [Cov91, HSSW98, CB03, HK15]). We do not need to assume that the investor knows the value of M .

2.2. Portfolio and relative value. A *portfolio vector* is an element of $\overline{\Delta}_n$, the closed unit simplex. In particular, all portfolios considered are fully invested in the stock market, and short selling is prohibited. At each time t the investor has to choose a portfolio vector $\pi(t)$, and the performance of the portfolio will be measured relative to the market portfolio. Formally, we define

$$V_\pi(t) = \frac{\text{growth of \$1 of the portfolio } \pi}{\text{growth of \$1 of the market portfolio } \mu}.$$

Definition 2.2 (Relative value). Let $\{\pi(t)\}_{t=0}^\infty$ be sequence of portfolio vectors. Given the market weight sequence $\{\mu(t)\}_{t=0}^\infty$, the relative value of π (with respect to the market portfolio) is the sequence $\{V_\pi(t)\}_{t=0}^\infty$ defined by $V_\pi(0) = 1$ and

$$(2.3) \quad \frac{V_\pi(t+1)}{V_\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)} =: \pi(t) \cdot \frac{\mu(t+1)}{\mu(t)}, \quad t \geq 0.$$

Here $a \cdot b$ is the Euclidean inner product and a/b is the vector of componentwise ratios whenever they are well-defined.

For a derivation of (2.3) see [PW13]. We will restrict to portfolio strategies that are deterministic functions of the current market weight, i.e., $\pi(t) = \pi(\mu(t))$. In this case a portfolio strategy is fully specified by a function $\pi : \Delta_n \rightarrow \overline{\Delta}_n$.

Definition 2.3 (Portfolio map). A portfolio map is a function $\pi : \Delta_n \rightarrow \overline{\Delta}_n$. The market portfolio is the identity map $\pi(p) = p$ and will be denoted by μ . A portfolio is said to be constant-weighted if π is identically constant.

2.3. Cover's portfolio as a market portfolio of portfolios. Let Θ be an index set and suppose each $\theta \in \Theta$ is associated with a portfolio map π_θ . The individual components of π_θ will be denoted by $(\pi_{\theta,1}, \dots, \pi_{\theta,n})$. (Sometimes we will use π_1, \dots, π_k to refer to a sequence of portfolios; the meaning should be clear from the context.) We are interested in the properties of $V_\theta(t) := V_{\pi_\theta}(t)$ as a function of *both* t and θ . To this end, we will consider an imaginary market whose basic assets are the portfolios π_θ .

We assume that Θ is a topological space and we are given a Borel probability measure ν_0 on Θ . The measure ν_0 will be called the *initial distribution*. The *support* $\text{supp}(\nu_0)$ of ν_0 is the smallest closed subset F of Θ satisfying $\nu_0(F) = 1$. We say that ν_0 has *full support* if $\text{supp}(\nu_0) = \Theta$. Intuitively, the imaginary market is defined by distributing unit wealth over the portfolios $\{\pi_\theta\}_{\theta \in \Theta}$ according to the initial distribution ν_0 , and letting the portfolios evolve. At time 0, the portfolio π_θ receives wealth $\nu_0(d\theta)$ which grows to $V_\theta(t)\nu_0(d\theta)$ at time t . Thus

$$(2.4) \quad \widehat{V}(t) := \int_{\Theta} V_\theta(t) d\nu_0(\theta)$$

is the total relative value of the imaginary market at time t . In order that (2.4) and related quantities (such as (2.5)) are well defined, we assume that the map $(p, \theta) \mapsto \pi_\theta(p)$ on $\Delta_n \times \Theta$ is jointly measurable in (p, θ) . This conditions usually follow immediately from the definition of the family considered. By Assumption 2.1, we have $V_\pi(t+1)/V_\pi(t) \leq M$ for any portfolio, so $V^*(t) < \infty$ and the integral in (2.4) is finite.

Definition 2.4 (Wealth distribution). Given a family of portfolios $\{\pi_\theta\}_{\theta \in \Theta}$ and an initial distribution ν_0 , the wealth distribution is the sequence of Borel probability measures $\{\nu_t\}_{t=0}^\infty$ defined by

$$(2.5) \quad \nu_t(B) = \frac{1}{\widehat{V}(t)} \int_B V_\theta(t) d\nu_0(\theta),$$

where B ranges over the measurable subsets of Θ .

Note that $\frac{d\nu_t}{d\nu_0}(\theta) = \frac{1}{\widehat{V}(t)} V_\theta(t)$. The main interest in the quantity $\widehat{V}(t)$ is the following fact first exploited by Cover in [Cov91], where $\{\pi_\theta\}_{\theta \in \Theta}$ is the family of constant-weighted portfolios. A proof can be found in [CB03, Lemma 3.1].

Lemma 2.5 (Cover's portfolio). *For each t , define the portfolio weight vector*

$$(2.6) \quad \hat{\pi}(t) := \int_{\Theta} \pi_{\theta}(\mu(t)) d\nu_t(\theta).$$

Then $V_{\hat{\pi}}(t) \equiv \hat{V}(t)$ for all t . We call $\hat{\pi}$ Cover's portfolio.

Let

$$(2.7) \quad V^*(t) = \sup_{\theta \in \Theta} V_{\theta}(t)$$

be the performance of the best portfolio in the family over the time interval $[0, t]$. The original goal of Cover's portfolio (2.6) is to track $V^*(t)$ in the sense that

$$(2.8) \quad \frac{1}{t} \log \frac{\hat{V}(t)}{V^*(t)} \rightarrow 0$$

as $t \rightarrow \infty$. If (2.8) holds, the portfolio $\hat{\pi}$ performs asymptotically as good as the best portfolio in the family. In Section 3.3 we give a simple example to show that (2.8) does not always hold. This question is naturally related to the concentration of the wealth distribution and motivates our study.

Remark 2.6. As pointed out by many authors (see for example [CB03]), the construction of Cover's portfolio (2.6) as a wealth-weighted average has a strong Bayesian flavor. Imagine the problem of finding the best portfolio in the family $\{\pi_{\theta}\}_{\theta \in \Theta}$. Little is known at time 0, but from historical data, experience and insider knowledge one may form a *prior distribution* ν_0 which describes the belief of the investor. At time t , having observed the returns of the portfolios up to time t , the investor updates the belief with the *posterior distribution* ν_t which satisfies

$$\frac{d\nu_t}{d\nu_0}(\theta) \propto \frac{V_{\theta}(t)}{V_{\theta}(0)} = V_{\theta}(t).$$

This corresponds to Bayes' rule where the relative return plays the role of the likelihood. Note that this procedure is time-consistent. Namely, for $t > s$, we have

$$\frac{d\nu_t}{d\nu_s}(\theta) \propto \frac{V_{\theta}(t)}{V_{\theta}(s)}.$$

Cover's portfolio (2.6) is then the posterior mean of $\pi_{\theta}(\mu(t))$.

3. LDP FOR TOTALLY BOUNDED FAMILIES

To gain intuition about how Cover's portfolio and the wealth distribution behave for a general (possibly nonparametric) family, and to prepare for the more technical treatment of functionally generated portfolio in Section 4, in this section we study large deviation properties of wealth distributions where the family of portfolios is totally bounded with respect to the uniform metric. We will use the following representation of portfolio value which is a direct consequence of Definition 2.2.

Lemma 3.1. *Let $\pi : \Delta_n \rightarrow \overline{\Delta}_n$ be a portfolio map. Then*

$$(3.1) \quad \frac{1}{t} \log V_{\pi}(t) = \int_{\Delta_n \times \Delta_n} \ell_{\pi}(p, q) d\mathbb{P}_t(p, q)$$

for all $t \geq 1$, where

$$(3.2) \quad \ell_{\pi}(p, q) := \log \left(\pi(p) \cdot \frac{q}{p} \right)$$

and

$$(3.3) \quad \mathbb{P}_t := \frac{1}{t} \sum_{s=0}^{t-1} \delta_{(\mu(s), \mu(s+1))}$$

is the empirical measure of the pair $(\mu(s), \mu(s+1))$ up to time t .

3.1. Finite state. To fix ideas we begin with an even simpler situation where the sequence $\{\mu(t)\}_{t=0}^\infty$ takes values in a *finite* set $E \subset \Delta_n$. The finite set E may be obtained by approximating the simplex by a finite grid.

Let

$$\Theta = \{\pi : E \rightarrow \overline{\Delta}_n\} = (\overline{\Delta}_n)^E$$

be the set of all portfolio maps on E . (Note that the family is indexed by the symbol π itself.) We equip Θ with the topology of uniform convergence. Since E is finite, this is the same as the topology of pointwise convergence. Note that Θ is compact and convex.

Lemma 3.2. *Suppose \mathbb{P}_t converges weakly to a probability measure \mathbb{P} on $E \times E$. Then for each $\pi \in \Theta$, the asymptotic growth rate exists and we have*

$$W(\pi) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V_\pi(t) = \int_{E \times E} \ell_\pi d\mathbb{P}.$$

Moreover, there is a portfolio $\pi^* \in \Theta$ satisfying

$$(3.4) \quad W(\pi^*) = W^* := \max_{\pi \in \Theta} W(\pi).$$

If we write $\mathbb{P}(p, q) = \mathbb{P}_1(p)\mathbb{P}_2(q | p)$, where \mathbb{P}_1 is the first marginal and \mathbb{P}_2 is the conditional distribution, then

$$(3.5) \quad \pi^*(p) = \arg \max_{x \in \overline{\Delta}_n} \int_E \log \left(x \cdot \frac{q}{p} \right) \mathbb{P}_2(dq | p)$$

for all p where $\mathbb{P}_1(p) > 0$.

A portfolio satisfying (3.4) may be called a *log-optimal portfolio*.

Proof. Since $E \times E$ is a finite set, by weak convergence we have

$$W(\pi) = \lim_{t \rightarrow \infty} \int_{E \times E} \ell_\pi d\mathbb{P}_t = \int_{E \times E} \ell_\pi d\mathbb{P}.$$

Thus the asymptotic growth rate exists for all $\pi \in \Theta$. Clearly $W(\cdot)$ is a continuous function on Θ . Since Θ is compact, it has a maximizer π^* . The last statement follows from the representation

$$W(\pi) = \int_E \left(\int_E \ell_\pi(p, q) \mathbb{P}_2(dq | p) \right) \mathbb{P}(dp).$$

□

The following LDP is a special case of Theorem 1.2 which will be proved in the next subsection.

Theorem 3.3 (Finite state LDP). *Suppose $\{\mu(t)\}_{t=0}^\infty$ takes values in a finite set $E \subset \Delta_n$. Let $\Theta = (\overline{\Delta}_n)^E$ and suppose that the initial distribution ν_0 has full support.*

(i) The portfolio $\hat{\pi}$ satisfies the universality property (3.6).

$$(3.6) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\hat{V}(t)}{V^*(t)} = 0.$$

(ii) If \mathbb{P}_t converges weakly to a probability measure \mathbb{P} on $E \times E$, the family $\{\nu_t\}_{t=0}^\infty$ satisfies the large deviation principle on Θ with the convex rate function

$$I(\pi) = W^* - W(\pi).$$

Remark 3.4. In the setting of Theorem 3.3(i), it is not difficult to show (see [CB03, Theorem 3.1]) that $V^*(t)/\hat{V}(t)$ is bounded above by a constant multiple of t^d , where $d = |E|(n-1)$ is the ‘dimension’ of Θ and $|E|$ is the cardinality of E .

3.2. LDP for totally bounded families. In this subsection we prove Theorem 1.2. Now $\{\mu(t)\}_{t=0}^\infty$ is any sequence in Δ_n satisfying Assumption 2.1.

Let Θ be a subset of $L^\infty(\Delta_n, \overline{\Delta}_n)$, the set of functions from Δ_n to $\overline{\Delta}_n$ equipped with the supremum metric $\|\cdot\|_\infty$ (defined in terms of the Euclidean norm $|\cdot|$ on $\overline{\Delta}_n$). We endow Θ with the induced topology, i.e., the topology of uniform convergence. A consequence of Assumption 2.1 is that the function $\ell_\pi(\cdot, \cdot)$ defined by (3.2) is bounded on \mathcal{S} between $\log \frac{1}{M}$ and $\log M$, for any $\pi \in \Theta$.

We say that Θ is *totally bounded* if for any $\epsilon > 0$, there exists $\pi_1, \dots, \pi_N \in \Theta$ with the following property: for any $\pi \in \Theta$, there exists $1 \leq j \leq N$ such that $\|\pi - \pi_j\|_\infty < \epsilon$. The smallest such N is called the ϵ -covering number of Θ . Thus Θ is totally bounded if and only if the covering number is finite for all $\epsilon > 0$.

First we prove a lemma which generalizes [CB03, Theorem 3.1] to nonparametric families. In this generality it seems that a quantitative bound like (1.3) is out of reach.

Lemma 3.5. *Suppose the market satisfies Assumption 2.1. Let Θ be a totally bounded subset of $L^\infty(\Delta_n, \overline{\Delta}_n)$ and let ν_0 be any initial distribution on Θ with full support. Then Cover’s portfolio $\hat{\pi}$ satisfies the universality property (3.6).*

Proof. It suffices to show that $\liminf_{t \rightarrow \infty} \frac{1}{t} \log \frac{\hat{V}(t)}{V^*(t)} \geq 0$. Let $\epsilon > 0$ be given. Then there exists $\epsilon' > 0$ and portfolios $\pi_1, \dots, \pi_N \in \Theta$ such that the set $\{\pi_j\}_{1 \leq j \leq N}$ are ϵ' -dense in Θ , and whenever $\|\pi - \pi_j\|_\infty < \epsilon'$ we have $|\ell_\pi - \ell_{\pi_j}| < \epsilon$ on \mathcal{S} .

For every $t > 0$, there exists a portfolio $\pi^{[t]} \in \Theta$ such that

$$\frac{1}{t} \log V_{\pi^{[t]}}(t) > \frac{1}{t} \log V^*(t) - \epsilon,$$

and from the above construction there exists $1 \leq j^{[t]} \leq N$ such that $\pi^{[t]} \in B(\pi_{j^{[t]}}, \epsilon')$, the open ball in Θ with radius ϵ' centered at $\pi_{j^{[t]}}$. Thus

$$(3.7) \quad \left| \frac{1}{t} \log V_{\pi_{j^{[t]}}}(t) - \frac{1}{t} \log V^*(t) \right| < 2\epsilon.$$

Moreover, if $\pi \in B_{j^{[t]}} := B(\pi_{j^{[t]}}, \epsilon')$, then

$$\left| \frac{1}{t} \log V_\pi(t) - \frac{1}{t} \log V_{\pi_{j^{[t]}}}(t) \right| < \epsilon$$

for all t . Combining these inequalities, we have

$$\begin{aligned}
 (3.8) \quad \frac{1}{t} \log \widehat{V}(t) &\geq \frac{1}{t} \log \int_{B_{j[t]}} V_\pi(t) \nu_0(d\pi) \\
 &\geq \frac{1}{t} \log \int_{B_{j[t]}} \exp \left(t \cdot \left(\frac{1}{t} \log V^*(t) - 3\epsilon \right) \right) \nu_0(d\pi) \\
 &\geq \frac{1}{t} \log V^*(t) - 3\epsilon + \frac{1}{t} \log \nu_0(B_{j[t]}).
 \end{aligned}$$

Since ν_0 has full support, we have $\lim_{t \rightarrow \infty} \min_{1 \leq j \leq N} \frac{1}{t} \log \nu_0(B_j) = 0$. Hence

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \frac{\widehat{V}(t)}{V^*(t)} \geq -3\epsilon.$$

The proof is completed by letting $\epsilon \rightarrow 0$. \square

Theorem 1.2 is a consequence Lemma 3.5 and the following ‘uniform strong law of large numbers’. The proof is a standard bracketing argument similar to the proof of Lemma 3.5 and can be found, for example, in [vdG00, Section 3.1].

Lemma 3.6. *Under the hypotheses of Theorem 1.2, we have*

$$\lim_{t \rightarrow \infty} \sup_{\pi \in \Theta} \left| \frac{1}{t} \log V_\pi(t) - W(\pi) \right| = 0.$$

Proof of Theorem 1.2. By assumption, $W(\pi) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V_\pi(t)$ exists for all $\pi \in \Theta$. Using the argument of the proof of Lemma 3.5, it can be shown that

$$(3.9) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \log \widehat{V}(t) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V^*(t) = W^*(t).$$

Since

$$\begin{aligned}
 \frac{1}{t} \log \nu_t(B) &= \frac{1}{t} \log \left(\frac{1}{\widehat{V}(t)} \int_{\Theta} V_\pi(t) d\nu_0(\pi) \right) \\
 &= \frac{1}{t} \log \left(\int_{\Theta} V_\pi(t) d\nu_0(\pi) \right) - \frac{1}{t} \log \widehat{V}(t)
 \end{aligned}$$

and thanks to (3.9), to prove the LDP it suffices to show that

$$(3.10) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log \int_F V_\pi(t) d\nu_0(\pi) \leq \sup_{\pi \in F} W(\pi)$$

for all closed sets F and

$$(3.11) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \log \int_G V_\pi(t) d\nu_0(\pi) \geq \inf_{\pi \in G} W(\pi)$$

for all open sets G . Indeed, we will show that (3.10) holds for all measurable sets no matter it is closed or not.

By Lemma 4.10, the quantity

$$R(t) = \sup_{\pi \in \Theta} \left| \frac{1}{t} \log V_\pi(t) - W(\pi) \right|$$

converges to 0 as $t \rightarrow \infty$. To prove the upper bound, write

$$\begin{aligned} \frac{1}{t} \log \int_F V_\pi(t) d\nu_0(\pi) &\leq \frac{1}{t} \log \int_F \exp(t(W(\pi) + R(t))) d\nu_0(\pi) \\ &\leq \sup_{\pi \in F} W(\pi) + R(t) + \frac{1}{t} \nu_0(F) \\ &\leq \sup_{\pi \in F} W(\pi) + R(t), \end{aligned}$$

since $\nu_0(F) \leq 1$ for all F . Letting $t \rightarrow \infty$ establishes the upper bound for all measurable sets. The lower bound for open sets can be proved in a similar manner using the fact that ν_0 has full support. \square

Proof of Theorem 3.3. Since E is finite, Θ is a totally bounded family of functions from E to $\overline{\Delta}_n$. (It can be extended from E to Δ_n by setting $\pi(p) = \pi_0$ for $p \notin E$, where π_0 is a fixed element of $\overline{\Delta}_n$.) The first statement then follows from Lemma 3.5. Moreover, by Lemma 3.2 the limit $W(\pi) = \lim_{t \rightarrow \infty} \int_{E \times E} \ell_\pi d\mathbb{P}_t$ exists and equals $\int_{E \times E} \ell_\pi d\mathbb{P}$ for all $\pi \in \Theta$. Thus Theorem 1.2 applies. It is easy to see that $I(\pi)$ is convex in π . \square

3.3. An example. Theorem 1.2 assumes that the family is totally bounded in the supremum metric and the asymptotic growth rates of all portfolios exist. Now we give a simple example to show what might go wrong. First, if the family is too large and the topology is not chosen appropriately, universality may fail. Second, the LDP may be trivial even if there is an optimal portfolio.

Consider a market with two stocks (so $n = 2$). Assume that the market weight takes values in the countable set

$$E = \{p = (p_1, p_2) \in \Delta_2 : p_1, p_2 \in \mathbb{Q}\}.$$

Let $\Theta = (\overline{\Delta}_2)^E$ be the set of portfolio maps on E and equip Θ with the topology of *pointwise convergence*. Let the initial distribution ν_0 be the infinite product of the uniform distribution on $\overline{\Delta}_2$. Then ν_0 has full support on Θ .

Let $\delta > 0$ be a rational number and consider the path $\{\mu(t)\}_{t \geq 0}$ in E defined recursively by

$$(3.12) \quad \mu(0) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad \mu(t+1) = \left(\frac{\mu_1(t)}{1 + \delta\mu_2(t)}, \frac{(1 + \delta)\mu_2(t)}{1 + \delta\mu_2(t)}\right).$$

Note that

$$(3.13) \quad \frac{\mu_2(t+1)}{\mu_2(t)} = (1 + \delta) \frac{\mu_1(t+1)}{\mu_1(t)}$$

for all $t \geq 0$ and it can be verified directly that $\{\mu(t)\}_{t \geq 0}$ satisfies Assumption 2.1 with $M = 1 + \delta$.

From (3.13), it is clear that any optimal portfolio π up to time t satisfies $\pi(\mu(s)) = (0, 1)$ for all $0 \leq s \leq t - 1$. It follows that

$$V^*(t) = \max_{\pi \in \Theta} V_\pi(t) = \frac{\mu_2(t)}{\mu_2(0)}$$

for all t .

Proposition 3.7. *For the market weight path given by (3.12), Cover's portfolio $\hat{\pi}$ satisfies*

$$\hat{V}(t) = \frac{\mu_2(t)}{\mu_2(0)} \left(1 - \frac{1}{2} \frac{\delta}{1+\delta} \right)^t.$$

In particular, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \frac{\hat{V}(t)}{V^*(t)} = \log \left(1 - \frac{1}{2} \frac{\delta}{1+\delta} \right) < 0.$$

Thus Cover's portfolio does not satisfy the universality property (3.6) for all market weight paths satisfying Assumption 2.1.

Proof. Given a portfolio $\pi \in \Theta$, we have

$$V_\pi(t) = \prod_{s=0}^{t-1} \left(\pi_1(\mu(s)) \frac{\mu_1(s+1)}{\mu_1(s)} + \pi_2(\mu(s)) \frac{\mu_2(s+1)}{\mu_2(s)} \right).$$

By (3.13), we can write

$$\begin{aligned} V_\pi(t) &= \prod_{s=0}^{t-1} \left(\frac{\mu_2(s+1)}{\mu_2(s)} \left(\frac{1}{1+\delta} \pi_1(\mu(s)) + \pi_2(\mu(s)) \right) \right) \\ &= \frac{\mu_2(t)}{\mu_2(0)} \prod_{s=0}^{t-1} \left(1 - (1 - \pi_2(\mu(s))) \frac{\delta}{1+\delta} \right). \end{aligned}$$

The value of Cover's portfolio is

$$\hat{V}(t) = \frac{\mu_2(t)}{\mu_2(0)} \int_{\Theta} \prod_{s=0}^{t-1} \left(1 - (1 - \pi_2(\mu(s))) \frac{\delta}{1+\delta} \right) d\nu_0(\pi).$$

Since ν_0 is an infinite product of uniform distributions, by independence we have

$$\hat{V}(t) = \frac{\mu_2(t)}{\mu_2(0)} \left(1 - \frac{1}{2} \frac{\delta}{1+\delta} \right)^t.$$

□

Proposition 3.8. *For the market weight path given by (3.12), the wealth distributions $\{\nu_t\}_{t=0}^\infty$ satisfies LDP on Θ with the trivial rate function $I(\pi) \equiv 0$.*

Proof. Let G be any open set of Θ . Then G contains a cylinder set of the form

$$(3.14) \quad C = \{(\pi(p_1), \dots, \pi(p_\ell)) \in B\},$$

where $p_1, \dots, p_\ell \in E$ and B is an open subset of $(\overline{\Delta}_2)^\ell$. It follows that

$$\nu_t(G) \geq \frac{1}{\left(1 - \frac{1}{2} \frac{\delta}{1+\delta}\right)^t} \int_C \prod_{s=0}^{t-1} \left(1 - (1 - \pi_2(\mu(s))) \frac{\delta}{1+\delta} \right) d\nu_0(\pi).$$

Using the fact that C puts restrictions on only finitely many coordinates, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \int_C \prod_{s=0}^{t-1} \left(1 - (1 - \pi_2(\mu(s))) \frac{\delta}{1+\delta} \right) d\nu_0(\pi) = \log \left(1 - \frac{1}{2} \frac{\delta}{1+\delta} \right).$$

So

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \nu_t(G) \geq 0$$

and $\lim_{t \rightarrow \infty} \frac{1}{t} \log \nu_t(G) = 0$. Since the upper bound holds trivially, the LDP is proved. \square

4. FUNCTIONALLY GENERATED PORTFOLIOS

This section is devoted to proving Theorem 1.3 for functionally generated portfolios. As in Section 3 we impose Assumption 2.1 on the market weight sequence $\{\mu(t)\}_{t=0}^\infty$. We begin by stating some properties of functionally generated portfolios defined in Section 1.3. For convex analytic concepts a standard reference is [Roc97].

4.1. Functionally generated portfolios. First we give the convex analytic interpretation of the defining inequality (1.8). Let $\Phi : \Delta_n \rightarrow \mathbb{R}$ be a concave function. The *superdifferential* $\partial\Phi(p)$ of Φ at $p \in \Delta_n$ is the convex set of all vectors $\xi \in \mathbb{R}^n$ satisfying $\sum_{i=1}^n \xi_i = 0$ (i.e., ξ is tangent to Δ_n) and

$$\Phi(p) + \langle \xi, q - p \rangle \geq \Phi(q)$$

for all $q \in \Delta_n$. The elements of $\partial\Phi(p)$ are called *supergradients* of Φ at p . Note that if Φ is positive and concave, $\log \Phi$ is also a concave function.

Lemma 4.1. [PW15, Proposition 6]

- (i) Suppose π is generated by Φ . For every $p \in \Delta_n$, the tangent vector $v = (v_1, \dots, v_n)$ of Δ_n given by

$$v_i = \frac{\pi_i(p)}{p_i} - \frac{1}{n} \sum_{j=1}^n \frac{\pi_j(p)}{p_j}$$

is an element of $\partial \log \Phi(p)$, the superdifferential of $\log \Phi$ at p .

- (ii) Conversely, suppose Φ is a positive concave function on Δ_n . For each $p \in \Delta_n$, let $v(p)$ be an element of $\partial \log \Phi(p)$ and define $\pi(p)$ by

$$\pi_i(p) = p_i \left(v_i(p) + 1 - \sum_{j=1}^n p_j v_j(p) \right).$$

Then π is a map from Δ_n to $\overline{\Delta}_n$ and is a portfolio generated by Φ .

If Φ is not differentiable at p , the superdifferential $\partial \log \Phi(p)$ is an infinite set, and by Lemma 4.1 there are multiple ways to choose a portfolio generated by Φ . Nevertheless, it is well known that a finite-valued concave function on Δ_n is differentiable almost everywhere on Δ_n , so the portfolios generated by Φ agree almost everywhere. Note, forever, that the null set depends on Φ . In general, a functionally generated portfolio $\pi : \Delta_n \rightarrow \overline{\Delta}_n$ is not continuous on Δ_n .

Let $\mathcal{FG} \subset L^\infty(\Delta_n, \overline{\Delta}_n)$ be the family of all functionally generated portfolios $\pi : \Delta_n \rightarrow \overline{\Delta}_n$. It is known that \mathcal{FG} is convex. Indeed, if π is generated by Φ and η is generated by Ψ , then for any $\lambda \in (0, 1)$ the portfolio $\lambda\pi + (1 - \lambda)\eta$ (a constant-weighted portfolio of π and η) is generated by the geometric mean $\Phi^\lambda \Psi^{1-\lambda}$. We endow \mathcal{FG} with the topology of uniform convergence. The following lemma shows that the current setting is not covered by Theorem 1.2.

Lemma 4.2. \mathcal{FG} is not totally bounded. In fact, \mathcal{FG} is not separable.

Proof. We give an example for $n = 2$ and similar considerations can be applied to all dimensions. For each $\theta \in (0, 1)$, let $\pi_\theta : \Delta_2 \rightarrow \overline{\Delta}_2$ be the portfolio

$$\pi_\theta(p) = \begin{cases} (1, 0) & \text{if } p_1 \leq \theta \\ (0, 1) & \text{if } p_1 > \theta. \end{cases}$$

It is easy to verify that each π_θ is functionally generated and $\{\pi_\theta\}_{\theta \in (0, 1)}$ forms an uncountable discrete set in \mathcal{FG} . Hence \mathcal{FG} is not separable. \square

Although the portfolio maps $\pi : \Delta_n \rightarrow \overline{\Delta}_n$ are the primary objects, it is technically more convenient to work with their generating functions.

Definition 4.3. Let \mathcal{C}_0 be the set of all positive concave functions Φ on Δ_n satisfying the normalization $\Phi(\bar{e}) = 1$, where $\bar{e} = (\frac{1}{n}, \dots, \frac{1}{n})$ is the barycenter of \bar{e} . We endow \mathcal{C}_0 with the topology of local uniform convergence. We define a metric d on \mathcal{C}_0 as follows. For $m = 1, 2, \dots$, let $K_m = \{p \in \Delta_n : p_i \geq \frac{1}{m}, 1 \leq i \leq n\}$. Then $\{K_m\}_{m=1}^\infty$ is a compact exhaustion of Δ_n . For $\Phi, \Psi \in \mathcal{C}_0$ we define

$$d(\Phi, \Psi) = \sum_{m=1}^\infty 2^{-m} \frac{\max_{p \in K_m} |\Phi(p) - \Psi(p)|}{1 + \max_{p \in K_m} |\Phi(p) - \Psi(p)|}.$$

By [PW15, Proposition 6] the generating function of a functionally generated portfolio is unique up to a positive multiplicative constant. Thus by a normalization we may assume without loss of generality that \mathcal{C}_0 is the set of generating functions.

Lemma 4.4. (\mathcal{C}_0, d) is a compact metric space.

Proof. See [Won15, Lemma 10]. \square

Although \mathcal{FG} is not totally bounded, by Lemma 4.1 and Lemma 4.4 it is ‘almost the same’ as \mathcal{C}_0 which is a compact metric space. This allows us to show under appropriate conditions that $V_\pi(t)$ behaves nicely as a function of π when t is large. Here is an application of the compactness of \mathcal{C}_0 .

Lemma 4.5. For each $t \geq 0$, there exists $\pi^* \in \Theta$ such that $V_{\pi^*}(t) = \sup_{\pi \in \mathcal{FG}} V_\pi(t)$.

Proof. The proof is essentially the one in [Won15, Theorem 4(i)] and is included for completeness. Let $\{\pi_k\}_{k=1}^\infty$ be a maximizing sequence, i.e.,

$$\sup_{\pi \in \mathcal{FG}} V_\pi(t) = \lim_{k \rightarrow \infty} V_{\pi_k}(t) = \lim_{k \rightarrow \infty} \prod_{s=0}^{t-1} \left(\pi_k(\mu(s)) \cdot \frac{\mu(s+1)}{\mu(s)} \right).$$

Let $\{\Phi_k\}_{k=1}^\infty \subset \mathcal{C}_0$ be the corresponding generating functions. By the compactness of \mathcal{C}_0 we may pass to a subsequence so that $\Phi_k \rightarrow \Phi \in \mathcal{C}_0$ locally uniformly on Δ_n . We may pass to a further subsequence such that the limit $\lim_{k \rightarrow \infty} \pi_k(\mu(s))$ exists in $\overline{\Delta}_n$ for all $0 \leq s \leq t-1$.

Let π^* be a portfolio generated by Φ which exists by Lemma 4.1. We claim that if we redefine π^* on $\{\mu(s) : 0 \leq s \leq t-1\}$ by setting

$$\pi^*(\mu(s)) = \lim_{k \rightarrow \infty} \pi_k(\mu(s))$$

for $0 \leq s \leq t-1$, then π^* is still generated by Φ and so is an element of Θ . By (1.8) it suffices to check that

$$(4.1) \quad \pi^*(\mu(s)) \cdot \frac{q}{\mu(s)} \geq \frac{\Phi(q)}{\Phi(\mu(s))}$$

for all $0 \leq s \leq t-1$ and $q \in \Delta_n$. Now since π_k is generated by Φ_k , we have

$$\pi_k(\mu(s)) \cdot \frac{q}{\mu(s)} \geq \frac{\Phi_k(q)}{\Phi_k(\mu(s))}.$$

Letting $k \rightarrow \infty$, we get (4.1) and so π^* is generated by Φ . The lemma follows by noting that $V_{\pi^*}(t) = \lim_{k \rightarrow \infty} V_{\pi_k}(t)$. \square

Continuing the statistical analogy (see Remark 2.6), the portfolio π^* may be viewed as the maximum likelihood estimator of the portfolio which maximizes the asymptotic growth rate $W(\pi) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V_\pi(t)$.

Lemma 4.6. *Let $\Phi_0 \in \mathcal{C}_0$ and $p_0 \in \Delta_n$. Let $K \subset \Delta_n$ be a compact set whose (relative) interior contains p_0 . Then for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $\Phi \in \mathcal{C}_0$, $\max_{p \in K} |\Phi(p) - \Phi_0(p)| < \delta$ and $|q - p_0| < \delta$, we have*

$$\partial \log \Phi(q) \subset \partial \log \Phi(p_0) + \epsilon \overline{B}(0, 1).$$

Proof. This is a uniform version of [Roc97, Theorem 24.5]. We will proceed by contradiction. If the statement is false, there exists $\epsilon_0 > 0$ such that the following holds. For every $k \geq 1$, there exists $\Phi_k \in \mathcal{C}_0$ and $p_k \in \Delta_n$ such that

$$\max_{p \in K} |\Phi_k(p) - \Phi_0(p)| < \frac{1}{k}, \quad |p_k - p_0| < \frac{1}{k}$$

and

$$\partial \log \Phi_k(p_k) \not\subset \partial \log \Phi(p_0) + \epsilon_0 \overline{B}(0, 1).$$

This contradicts [Roc97, Theorem 24.5] and thus the lemma is proved. \square

Using Lemma 4.6 and Proposition 4.1 we have the following corollary which is a refined version of [Won15, Lemma 11].

Lemma 4.7. *Let π_0 be a portfolio generated by Φ_0 . Let $p_0 \in \Delta_n$ be a point at which Φ_0 is differentiable. For any $\epsilon > 0$ and any compact neighborhood K of p_0 in Δ_n , there exists $\delta > 0$ such that whenever π is generated by Φ and $\max_{p \in K} |\Phi(p) - \Phi_0(p)| < \delta$, we have $\max_{p: |p - p_0| < \delta} |\pi(p) - \pi_0(p_0)| < \epsilon$.*

We end this subsection with some technical remarks.

Remark 4.8. It is natural to ask why we do not use the compact set \mathcal{C}_0 as the index space. There are three reasons for this. First, the portfolio maps $\pi : \Delta_n \rightarrow \overline{\Delta}_n$ are the primary objects for portfolio analysis, and the generating functions are only derived entities. Second, even if π_1 and π_2 have the same generating function, over a finite horizon $V_{\pi_1}(t)$ and $V_{\pi_2}(t)$ may have quite different behaviors. Third, even though for each $\Phi \in \mathcal{C}_0$ we may choose a portfolio π_Φ generated by Φ , there is no canonical way of doing this so that the maps $\Phi \mapsto \pi_\Phi$ and $\Phi \mapsto V_{\pi_\Phi}(t)$ are measurable.

4.2. Asymptotic growth rate. Recall from Lemma 3.1 that $V_\pi(t)$ can be written in the form

$$\frac{1}{t} \log V_\pi(t) = \int_S \ell_\pi d\mathbb{P}_t,$$

where

$$\ell_\pi(p, q) = \log \left(\pi(p) \cdot \frac{q}{p} \right)$$

is defined in (3.2) and

$$\mathbb{P}_t = \frac{1}{t} \sum_{s=0}^{t-1} \delta_{(\mu(s), \mu(s+1))}$$

is the empirical measure of the pair $(\mu(s), \mu(s+1)) \in \mathcal{S}$ up to time t . Appealing to the long term stability of capital distribution, we assume that \mathbb{P} converges weakly to an absolutely continuous probability measure \mathbb{P} . We denote by $B(p, \delta)$ the Euclidean ball in Δ_n centered at p with radius δ . The Euclidean norm is denoted by $|\cdot|$.

First we prove a ‘strong law of large numbers’ for individual elements of \mathcal{FG} . We will use some basic results of weak convergence theory [Bil09]. Recall that a \mathbb{P} -continuity set is a set A satisfying $\mathbb{P}(\partial A) = 0$, where ∂A is the boundary of A . We write $\partial_{\mathcal{S}} A$ if we want to be explicit about the underlying topological space.

Lemma 4.9. *Suppose \mathbb{P}_t converges weakly to an absolutely continuous probability measure \mathbb{P} on \mathcal{S} . Then for every $\pi \in \mathcal{FG}$ the asymptotic growth rate $W(\pi) = \lim_{t \rightarrow \infty} \frac{1}{t} \log V_{\pi}(t)$ exists and is given by*

$$(4.2) \quad W(\pi) = \lim_{t \rightarrow \infty} \int_{\mathcal{S}} \ell_{\pi} d\mathbb{P}_t = \int_{\mathcal{S}} \ell_{\pi} d\mathbb{P}.$$

Proof. Note that (4.2) does not follow directly from the definition of weak convergence because ℓ_{π} may have discontinuities. The constructions here (refined from the proof of [Won15, Theorem 5]) will be useful when we prove uniform convergence in Lemme 4.10.

Let $\epsilon > 0$ be given. Let $\Phi \in \mathcal{C}_0$ be the generating function of π and consider the set

$$D = \{p \in \Delta_n : \Phi \text{ is differentiable at } p\}.$$

Then $\Delta_n \setminus D$ has Lebesgue measure 0. Given ϵ , there exists $\epsilon' > 0$ such that whenever $\pi_1, \pi_2 \in \overline{\Delta}_n$ and $|\pi_1 - \pi_2| < \epsilon'$, we have

$$(4.3) \quad \left| \log \left(\pi_1 \cdot \frac{q}{p} \right) - \log \left(\pi_2 \cdot \frac{q}{p} \right) \right| < \epsilon$$

for all $(p, q) \in \mathcal{S}$.

For each $p \in D$, by Lemma 4.7 there exists $\delta(p) > 0$ such that $B(p, \delta(p)) \subset \Delta_n$ and $|q - p| < \delta(p)$ implies

$$(4.4) \quad |\pi(q) - \pi(p)| < \epsilon'.$$

As a subspace of a separable metric space, D is separable. Hence, there exists a countable set $\{p_k\}_{k=1}^{\infty} \subset D$ such that

$$D \subset \bigcup_{k=1}^{\infty} B(p_k, \delta(p_k)).$$

Let $A_1 = B(p_1, \delta(p_1))$ and for $k \geq 2$ define

$$A_k = B(p_k, \delta(p_k)) \setminus \bigcup_{j=1}^{k-1} B(p_j, \delta(p_j)).$$

Then the sets $\{A_k\}$ are disjoint and

$$(D \times \Delta_n) \cap \mathcal{S} \subset \bigcup_{k=1}^{\infty} (A_k \times \Delta_n) \cap \mathcal{S}.$$

Since $\mathbb{P}((D \times \Delta_n) \cap \mathcal{S}) = 1$ by absolute continuity, by continuity of measure there exists a positive integer k_0 such that

$$\mathbb{P}\left(\bigcup_{k=1}^{k_0} (A_k \times \Delta_n) \cap \mathcal{S}\right) > 1 - \epsilon.$$

Let

$$A_0 = \Delta_n \setminus \left(\bigcup_{k=1}^{k_0} A_k\right).$$

Then

$$(4.5) \quad \mathbb{P}((A_0 \times \Delta_n) \cap \mathcal{S}) \leq \epsilon.$$

Note that for $0 \leq k \leq k_0$, $(A_k \times \Delta_n) \cap \mathcal{S}$ is a \mathbb{P} -continuity set as it is formed by set-theoretic operations on \mathcal{S} (which has piecewise smooth boundary), Δ_n and Euclidean balls. Also, by Assumption 2.1 $|\ell_\pi(\cdot, \cdot)|$ is bounded uniformly on \mathcal{S} by $M' := \log M$. So, for each $1 \leq k \leq k_0$ the map

$$(p, q) \mapsto \ell_{\pi(p(k))}(p, q) := \log\left(\pi(p(k)) \cdot \frac{q}{p}\right)$$

is a bounded continuous function on \mathcal{S} .

By weak convergence and Lemma A.2 in the Appendix, there exists a positive integer t_0 such that for $t \geq t_0$ we have

$$(4.6) \quad \mathbb{P}_t((A_0 \times \Delta_n) \cap \mathcal{S}) < 2\epsilon$$

and

$$(4.7) \quad \left| \int_{(A_k \times \Delta_n) \cap \mathcal{S}} \ell_{\pi(p(k))} d(\mathbb{P}_t - \mathbb{P}) \right| < \frac{\epsilon}{k_0}.$$

(note that k_0 is fixed before t_0 is chosen).

Now we estimate the difference $\left| \frac{1}{t} \log V_\pi(t) - \int_{\mathcal{S}} \ell_\pi d\mathbb{P} \right| = \left| \int_{\mathcal{S}} \ell_\pi d(\mathbb{P}_t - \mathbb{P}) \right|$. We have

$$(4.8) \quad \left| \int_{\mathcal{S}} \ell_\pi d(\mathbb{P}_t - \mathbb{P}) \right| \leq \left| \sum_{k=1}^{k_0} \int_{(A_k \times \Delta_n) \cap \mathcal{S}} \ell_\pi d(\mathbb{P}_t - \mathbb{P}) \right| + \left| \int_{(A_0 \times \Delta_n) \cap \mathcal{S}} \ell_\pi d(\mathbb{P}_t - \mathbb{P}) \right|.$$

Using the boundedness of ℓ_π , (4.5) and (4.6), the second term of (4.8) is bounded by $3M'\epsilon$. Now for each k , by (4.3), (4.4) and (4.7) we have

$$\begin{aligned} \left| \int_{(A_k \times \Delta_n) \cap \mathcal{S}} \ell_\pi d(\mathbb{P}_t - \mathbb{P}) \right| &\leq \int_{(A_k \times \Delta_n) \cap \mathcal{S}} |\ell_\pi - \ell_{\pi(p_k)}| d\mathbb{P}_t \\ &\quad + \int_{(A_k \times \Delta_n) \cap \mathcal{S}} |\ell_\pi - \ell_{\pi(p_k)}| d\mathbb{P} \\ &\quad + \left| \int_{(A_k \times \Delta_n) \cap \mathcal{S}} \ell_{\pi(p_k)} d(\mathbb{P}_t - \mathbb{P}) \right| \\ &\leq \epsilon \mathbb{P}_t((A_k \times \Delta_n) \cap \mathcal{S}) + \epsilon \mathbb{P}((A_k \times \Delta_n) \cap \mathcal{S}) \\ &\quad + \frac{\epsilon}{k_0}. \end{aligned}$$

Summing the above inequality over k , we get

$$\left| \int_S \ell_\pi d(\mathbb{P}_t - \mathbb{P}) \right| \leq \epsilon + \epsilon + \epsilon + 3M'\epsilon, \quad t \geq t_0,$$

and the lemma is proved. \square

4.3. Glivenko-Cantelli property. Now we observe that the proof of Lemma 4.9 can be modified to yield a uniform version which implies Theorem 1.3(i). Recall that $d(\Phi, \Psi)$ is the metric on \mathcal{C}_0 given in Definition 4.3.

Lemma 4.10. *Suppose \mathbb{P}_t converges weakly to an absolutely continuous probability measure \mathbb{P} on \mathcal{S} . Let $\pi_0 \in \mathcal{FG}$ be generated by $\Phi_0 \in \mathcal{C}_0$. For any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$(4.9) \quad \limsup_{t \rightarrow \infty} \sup_{\pi \in \mathcal{FG}(\pi_0, \delta)} \left| \frac{1}{t} \log V_\pi(t) - \frac{1}{t} \log V_{\pi_0}(t) \right| < \epsilon,$$

where $\mathcal{FG}(\pi_0, \delta)$ is the set of all functionally generated portfolio π whose generating function $\Phi \in \mathcal{C}_0$ satisfies $d(\Phi, \Phi_0) < \delta$. In particular, we have the ‘uniform strong law of large numbers’

$$(4.10) \quad \lim_{t \rightarrow \infty} \sup_{\pi \in \mathcal{FG}} \left| \frac{1}{t} \log V_\pi(t) - W(\pi) \right| = 0.$$

Proof. We want to estimate

$$\sup_{\pi \in \mathcal{FG}(\pi_0, \delta)} \left| \frac{1}{t} \log V_\pi(t) - \frac{1}{t} \log V_{\pi_0}(t) \right| = \sup_{\pi \in \mathcal{FG}(\pi_0, \delta)} \left| \int_S (\ell_\pi - \ell_{\pi_0}) d\mathbb{P}_t \right|.$$

Recall from Definition 4.3 that $K_m = \{p \in \Delta_n : p_i \geq \frac{1}{m}\}$. By continuity of measure, we can choose m so that

$$\mathbb{P}((K_m \times \Delta_n) \cap S) > 1 - \epsilon.$$

Since $(K_m \times \Delta_n) \cap \mathcal{S}$ is a \mathbb{P} -continuity set, for t sufficiently large we have

$$\left| \left(\int_S - \int_{(K_m \times \Delta_n) \cap S} \right) (\ell_\pi - \ell_{\pi_0}) d\mathbb{P}_t \right| < 4M\epsilon,$$

where $M' = \log M$ is the upper bound of $|\ell_\pi|$ and $|\ell_{\pi_0}|$ on \mathcal{S} . This allows us to focus on the set $(K_m \cap \Delta_n) \cap \mathcal{S}$.

Fix $\epsilon' > 0$. By Lemma 4.7, for each p in the (relative) interior of K_m at which Φ_0 is differentiable (call this set D_m), there exists $\delta'(p) > 0$ such that whenever $\max_{q \in K_m} |\Phi(q) - \Phi_0(q)| < \delta'(p)$ and $|q - p| < \delta'(p)$, we have $|\pi(q) - \pi_0(p)| < \epsilon'$.

As in the proof of Lemma 4.9, we may cover D_m by a disjoint countable union $\bigcup_{k=1}^\infty A_k$, where A_k is a \mathbb{P} -continuity set containing p_k and has diameter bounded by $\delta'(p_k)$.

Now choose a positive integer k_0 such that

$$\mathbb{P} \left(\left(\bigcup_{k=1}^{k_0} A_k \times \Delta_n \right) \cap \mathcal{S} \right) > 1 - 2\epsilon.$$

Also, choose $\delta > 0$ such that

$$d(\Phi, \Phi_0) < \delta \Rightarrow \max_{p \in K_m} |\Phi(p) - \Phi_0(p)| < \min_{1 \leq k \leq k_0} \delta'(p_k).$$

It follows that

$$\sup_{\pi \in \mathcal{FG}(\pi_0, \delta(p_k))} \sup_{p: |p-p_k| < \delta'(p_k)} |\pi(p) - \pi_0(p_k)| < \epsilon',$$

With this uniform local approximation, we may follow the same steps as the proof of Lemma 4.9 to prove that

$$\limsup_{t \rightarrow \infty} \sup_{\pi \in \mathcal{FG}(\pi_0, \delta)} \left| \frac{1}{t} \log V_\pi(t) - W(\pi) \right| < C\epsilon,$$

where $C > 0$ is a constant. Thus (4.9) follows by letting $\epsilon \rightarrow 0$.

Note that (4.9) implies that

$$\sup_{\pi \in \mathcal{FG}(\pi_0, \delta)} |W(\pi) - W(\pi_0)| \leq \epsilon.$$

Since \mathcal{C}_0 is compact, we may cover \mathcal{FG} by finitely many sets of the form $\mathcal{FG}(\pi_0, \delta)$, and (4.10) follows. \square

4.4. LDP and universality. Now we finish the proof of Theorem 1.3. Recall that $\hat{V}(t) = \int_{\Theta} V_\pi(t) d\nu_0(\pi)$ and $V^*(t) = \sup_{\pi \in \Theta} V_\pi(t)$.

Lemma 4.11. *Suppose \mathbb{P}_t converges weakly to an absolutely continuous probability measure \mathbb{P} on \mathcal{S} . Let ν_0 be any initial distribution on \mathcal{FG} and $W^* = \sup_{\pi \in \text{supp}(\nu_0)} W(\pi)$. Then $\lim_{t \rightarrow \infty} \frac{1}{t} \log \hat{V}(t) = W^*$.*

Proof. For $\pi \in \mathcal{FG}$ write

$$\frac{1}{t} \log V_\pi(t) = W(\pi) + R_\pi(t)$$

where $R_\pi(t)$ is the remainder. By Lemma 4.10 we have $\lim_{t \rightarrow \infty} \sup_{\pi \in \mathcal{FG}} |R_\pi(t)| = 0$. Write

$$\hat{V}(t) = \int_{\text{supp}(\nu_0)} e^{t(W(\pi) + R_\pi(t))} d\nu_0(\pi).$$

It is clear that $\limsup_{t \rightarrow \infty} \frac{1}{t} \log \hat{V}(t) \leq W^*$. To show the other inequality, note that $W(\pi)$ is continuous in $\pi \in \mathcal{FG}$. Thus for any $\pi \in \text{supp}(\nu_0)$ and $\epsilon > 0$, by restricting the integral to a neighborhood of π we have $\liminf_{t \rightarrow \infty} \frac{1}{t} \log \hat{V}(t) \geq W(\pi) - \epsilon$. Taking supremum over $\pi \in \text{supp}(\nu_0)$ completes the proof. \square

Proof of Theorem 1.3. (i) This has been proved in Lemma 4.10.

(ii) We argue as in the proof of Theorem 1.3. Write

$$\nu_t(B) = \frac{1}{\hat{V}(t)} \int_{B \cap \text{supp}(\nu_0)} V_\pi(t) d\nu_0(\pi).$$

Using the uniform convergence property (i), we can show that

$$(4.11) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log \int_F V_\pi(t) d\nu_0(\pi) \leq \sup_{\pi \in F \cap \text{supp}(\nu_0)} W(\pi)$$

for any set F with $F \cap \text{supp}(\nu_0) \neq \emptyset$, and

$$(4.12) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \log \int_G V_\pi(t) d\nu_0(\pi) \geq \inf_{\pi \in G \cap \text{supp}(\nu_0)} W(\pi)$$

for all open sets G such that $G \cap \text{supp}(\nu_0) \neq \emptyset$. These inequalities and Lemma 4.11 imply the LDP.

(iii) Let $\{\Phi_k\}_{k=1}^\infty$ be a countable dense set in the metric space (\mathcal{C}_0, d) . For each k , let π_k be a portfolio generated by Φ_k . Consider an initial distribution of the form

$$(4.13) \quad \nu_0 = \sum_{k=1}^{\infty} \lambda_k \delta_{\pi_k},$$

where $\lambda_k > 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$.

To see that ν_0 works, let π be any functionally generated portfolio and $\Phi \in \mathcal{C}_0$ be its generating function. Then there is a sequence $\pi_{k'}$ whose generating functions $\Phi_{k'}$ converges locally uniformly to Φ . By Lemma 4.10, we have $W(\pi_{k'}) \rightarrow W(\pi)$. Thus $W^* = \sup_{\pi \in \text{supp}(\nu_0)} W(\pi) = \sup_{\pi \in \mathcal{FG}} W(\pi)$. By Lemma 4.11, to establish the asymptotic universality property (1.10) it remains to show that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log V^*(t) = W^*,$$

but this is a direct consequence of the uniform convergence property (i). \square

5. CONCLUSION AND FURTHER PROBLEMS

In this paper we studied Cover's portfolio from the point of view of stochastic portfolio theory. Given a family of portfolios, we studied its wealth distribution which is analogous to the capital distribution of an equity market. In this setting, the wealth distribution is not stable and diverse in the sense of stochastic portfolio theory, and under certain conditions we quantified its concentration in terms of large deviation principles. We also extended Cover's portfolio to the nonparametric family of functionally generated portfolios and established its asymptotic universality in the spirit of [Jam92].

Similar to [Jam92] and [GLU06], the results in this paper are asymptotic in nature, and in this nonparametric setting we are unable to establish quantitative bounds that hold for all finite horizons. It is desirable to obtain quantitative bounds despite of the fact that they may be too conservative to be useful in practice. Even if the underlying market process is modeled correctly, the convergence $\frac{1}{t} \log V_\pi(t) \rightarrow W(\pi)$ may take a long time and the portfolio $\hat{\pi}(t)$ may be dominated by noise. A possible remedy is to use a smaller family or to impose regularization via a suitable prior (initial distribution). Tackling this bias-variance trade-off in dynamic portfolio selection is an interesting problem of great practical importance.

Problem 5.1. For Cover's portfolio for the family of functionally generated portfolios, is it possible to choose an initial distribution such that $\hat{\pi}$ can be computed or approximated numerically and a quantitative lower bound of $\hat{V}(t)/V^*(t)$ can be proved?

Instead of using Cover's portfolio as a wealth-weighted average, we may use other portfolio selection algorithms to construct universal portfolios for functionally generated portfolios. Perhaps the *follow-the-regularized-leader* (FTRL) approach of [HK15] can be generalized to this nonparametric set up.

A classic result in asymptotic parametric statistics is the *Bernstein von-Mises Theorem* which states that the posterior distribution is asymptotically normal under appropriate scaling [VdV00, Chapter 10]. Certain generalizations to nonparametric models are possible, see for example [CN13]. As noted in the Introduction, for constant-weighted portfolios the map $\pi \mapsto V_\pi(t)$ is essentially a multiple of a normal

density (see [Jam92] and [CB03]). Hence the wealth distribution, when suitably rescaled, is approximately normal if the initial distribution is sufficiently regular. Since the family of functionally generated portfolios is convex, it can be viewed as an infinite dimensional constant-weighted family of portfolios.

Problem 5.2. Formulate and prove a version of Bernstein von-Mises Theorem in the setting of Theorem 1.3.

APPENDIX A.

The following lemmas are both standard results. Since we are unable to find suitable references, we will provide the proofs for completeness.

Lemma A.1. *Let X be a topological space and Y be a subset of X equipped with the subspace topology. If $A \subset Y$, then*

$$\partial_X A \subset \partial_Y A \cup \partial_X Y.$$

Proof. We will argue by contradiction. Suppose $x \in \partial_X A$ and $x \notin \partial_Y A \cup \partial_X Y$.

By the definition of subspace topology and boundary, there exist neighborhoods U_1 and U_2 of x in X such that

$$(1) U_1 \cap Y \subset A \quad \text{or} \quad (2) U_1 \cap Y \subset Y \setminus A,$$

and

$$(i) U_2 \subset Y \quad \text{or} \quad (ii) U_2 \subset X \setminus Y.$$

We may replace U_1 and U_2 above by their intersection $U = U_1 \cap U_2$. Also, since $x \in \partial_X A$, U intersects both A and $X \setminus A$. We claim that the above statements are incompatible. We consider the following cases.

(1) and (i): Since $U \subset Y$ and $U \cap Y \subset A$, we have $U \subset A$. This contradicts the fact that U intersects $X \setminus A$.

(2) and (i): We have $U \subset Y \setminus A$. But $A \subset Y$, so U does not intersect A and we have a contradiction.

(ii): If $U \cap Y = \emptyset$, then U does not intersect A which is a contradiction. \square

Lemma A.2. *Suppose \mathbb{P}_t converges weakly to \mathbb{P} . Let $f : \mathcal{S} \rightarrow \mathbb{R}$ be bounded continuous and let Y be a \mathbb{P} -continuity set in \mathcal{S} with $\mathbb{P}(Y) > 0$. Then*

$$\lim_{t \rightarrow \infty} \int_Y f d\mathbb{P}_t = \int_Y f d\mathbb{P}.$$

Proof. Consider the measures conditioned on Y :

$$\tilde{\mathbb{P}}_t(\cdot) = \frac{\mathbb{P}_t(\cdot \cap Y)}{\mathbb{P}_t(Y)}, \quad \tilde{\mathbb{P}}(\cdot) = \frac{\mathbb{P}(\cdot \cap Y)}{\mathbb{P}(Y)}.$$

Since $\mathbb{P}_t(Y) \rightarrow \mathbb{P}(Y) > 0$ as A is a \mathbb{P} -continuity set, the measures $\tilde{\mathbb{P}}_t$ are well defined for t sufficiently large.

We claim that $\tilde{\mathbb{P}}_t$ converges weakly to $\tilde{\mathbb{P}}$. This implies the statement because f is bounded continuous on Y and

$$\int_{\mathcal{S}} f d\tilde{\mathbb{P}}_t = \frac{1}{\mathbb{P}_t(Y)} \int_Y f d\mathbb{P}_t \rightarrow \frac{1}{\mathbb{P}(Y)} \int_Y f d\mathbb{P} = \int_{\mathcal{S}} f d\tilde{\mathbb{P}}.$$

To prove the claim, it suffices by the Portmanteau theorem to show that $\tilde{\mathbb{P}}_t(A) \rightarrow \tilde{\mathbb{P}}(A)$ for all $A \subset Y$ with $\tilde{\mathbb{P}}(\partial_Y A) = \frac{1}{\mathbb{P}(Y)} \mathbb{P}(\partial_Y A \cap Y) = 0$. Note that $\partial_Y A \subset Y$, so $\mathbb{P}(\partial_Y A) = 0$. By Lemma A.1, we have $\partial_S A \subset \partial_Y A \cup \partial_S Y$, and so $\mathbb{P}(\partial_S A) = 0$

as Y is a \mathbb{P} -continuity set. Thus $A = A \cap Y$ is a \mathbb{P} -continuity set and we have $\mathbb{P}_t(A) \rightarrow \mathbb{P}(A)$. This completes the proof of the lemma. \square

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